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Formation of two-dimensional incommensurate structures in the system of Williams domains

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The formation of incommensurate structures and their breakdown in the system of domains is shown with the help of a generalized thermodynamic potential describing Williams domains in nematic liquid crystals above the electrohydrodynamic instability threshold.

1. Introduction

On the application of an external electric field, the appearance of stationary non-equilibrium dissipative structures is possible in a nematic liquid crystal. An example of this fact may be Williams domains, a system of rotating hydrodynamic shafts which result in a liquid-crystalline film because of an electrohydrodynamic instability. The behaviour of the system in the electrohydrodynamic instability mode is essentially non-linear, and to describe this system various perturbation theories have been developed, which allow us to describe various aspects of this phenomenon not only qualitatively but to a considerable extent also quantitatively.

However the liquid crystal, being a non-linear, anisotropic medium, appears to be sufficiently rich in physical phenomena many of which are far from having been explained. Particularly, in the recently published experimental study [1], a two-dimensional pattern of distortion of electrohydrodynamic-rolls was observed, whose appearance, in the first view, contradicts the symmetry of an initial system. As we shall demonstrate, the appearance of structures having such symmetry becomes possible in the presence of an external electric field. Since there is no hope to solve precisely a complete set of non-linear differential equations in partial derivatives which describe the behaviour of a liquid crystal under various conditions, then success depends to a great extent on a good 'coarsening' of the initial system.

In the present study an approach is developed, based on the changeover from the initial set of differential equations to the so-called generalized thermodynamic potential whose minimum is realized in the most probable dissipative structures. A similar approach has been utilized previously in the hydrodynamics of isotropic liquids, in the Benard problem [2].

By applying this approach developed in [3], in which the variables introduced by the authors of the present paper act as the order parameter, it has become possible to find new spatially-periodic structures in the system of electrohydrodynamic shafts. Thus, the experimental observation described in [1] obtains a natural explanation within the framework of this approach.

2. Generalized thermodynamic potential

In this section we shall describe briefly the scheme used to obtain the functional [3]. As a starting point, we shall take the set of equations which describe a liquid crystal in the hydrodyamic approximation [3], namely

$$\left.\begin{aligned} J\frac{d\boldsymbol{\Omega}}{dt} &= \mathbf{n}\times\mathbf{h}-\boldsymbol{\Gamma},\\ \varrho d\mathbf{V}/dt &= \mathbf{f}\times\mathbf{g},\\ \operatorname{div}\mathbf{D} &= 4\pi q,\\ dq/dt+\operatorname{div}\mathbf{j} &= 0,\\ \operatorname{rot}\mathbf{E} &= 0. \end{aligned}\right\} \tag{1}$$

Here, J is the moment of inertia per unit volume for the nematic, $\boldsymbol{\Omega}$ is the angular speed of the director $\mathbf{n}$, $\mathbf{f}$ and $\mathbf{g}$ are forces caused by the maxwellian stress tensor and the liquid crystal stress tensor, respectively, $\mathbf{h}$ is the molecular field, $\boldsymbol{\Gamma}$ is the moment of the forces, $\mathbf{V}$ is the liquid crystal hydrodynamic velocity, q is the charge density, $\mathbf{j}$ is the current density and ϱ is the liquid crystal density. Let us assume a film of nematic liquid crystal having a thickness l is placed in the XOY plane; normal to this plane (along the OZ axis), an external electric field is applied, having in intensity E_0. For the purpose of definiteness, we shall consider that in an unperturbed state, the director is aligned (due to the orienting effect of the glass surface) along the OX axis. It is known from experiment that with a certain threshold value of the electric field, the liquid crystal loses its mechanical equilibrium, i.e. the electrohydrodynamic instability is developed. Our aim is to describe the phenomena occurring near (but above) the instability threshold.

In the investigation of the set of equations (1), it is convenient to introduce the vector

$$\mathbf{u} = (\mathbf{v}, \mathbf{n}, \mathbf{E}) \tag{2}$$

and to write the system in the symbolic form

$$\hat{\mathbf{L}}[\mathbf{u}] = 0 \tag{3}$$

The matrix form of the differential operator $\hat{\mathbf{L}}$ is found directly from the comparison between equations (1) and (3), taking into account (2). In the course of finding a solution to equation (3) near the threshold, we shall make use of perturbation theory, having taken the solution to the linearized set (1)

$$\hat{\mathbf{L}}_0[\mathbf{u}_0] = 0 \tag{4}$$

as an initial one. The solution to the linearized set

$$\mathbf{u}_0(t) = \mathbf{u}_0\exp(-\lambda t), \tag{5}$$

is easily found in complete analogy with [2]. In so doing, it turns out that among the damped modes there appears a mode having the eigenvalue

$$\lambda_1 = -\varepsilon\lambda_s. \tag{6}$$

Here, λ_s is the eigenvalue in the absence of the external field, E_0, and

$$\varepsilon \equiv \frac{R-R_c}{R_c}, \quad R = \frac{\varrho l^2}{\alpha_0^2}E_0^2, \tag{7}$$

where α_0 is the coefficient of viscosity. We can see that the eigenvalue of this mode, with $E_0 \to E_c$, tends to zero, which corresponds to the appearance of the electrohydrodynamic instability. We shall develop the perturbation theory by taking the magnitude ε as a small parameter.

The simplest version of such theory would consist of the expansion of the kind

$$\mathbf{u}(t) = \mathbf{u}_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots.$$

Another version of the perturbation theory could however be developed, by taking into account the fact that a variation of R by a magnitude εR changes (as already follows from the analysis of the linear theory) the corresponding sizes of typical fluctuations along the OX axis by a factor $1/\varepsilon$, along the OY axis, by $1/\sqrt{\varepsilon}$, and in terms of time, by a factor $1/\varepsilon^2$. Then following [2], it is expedient to introduce formally additional variables

$$x = \frac{\xi}{\varepsilon}, \quad y = \frac{\eta}{\sqrt{\varepsilon}}, \quad t = \frac{\tau}{\varepsilon^2} \tag{8}$$

and to carry out the substitution

$$\partial_x \to \partial_x + \varepsilon\partial_\xi, \quad \partial_y \to \partial_y + \sqrt{\varepsilon}\partial_\eta, \quad \partial_t \to \varepsilon^2\partial_\tau. \tag{9}$$

Then, having represented the operator $\hat{\mathbf{L}}$ in the series

$$\hat{\mathbf{L}} = \hat{\mathbf{L}}_0 + \varepsilon^{1/2}\hat{\mathbf{L}}_{1/2} + \varepsilon\hat{\mathbf{L}}_1 + \varepsilon^{3/2}\hat{\mathbf{L}}_{3/2} + \varepsilon^2\hat{\mathbf{L}}_2 + \cdots \tag{10}$$

and, similarly, the vector $\mathbf{u}(t)$ and, by equating to zero the terms at identical powers of ε, we shall obtain an improved version of the perturbation theory.

The solution to the non-linear system will be sought in the form

$$\mathbf{u} = W\mathbf{u}_0 + W^*\mathbf{u}_0^*. \tag{11}$$

The representation of the solution in a predetermined form, taking into account the procedure for its generation, is equivalent to the widely utilized method of shortened equations in the problems of non-linear optics. As accepted in hydrodynamics problems (when studying instability), we shall obtain an equation for the amplitude W which also accounts for, apart from non-linear terms, random noise (thermal fluctuations). By restricting to the lowest non-linearity in terms of the amplitude W, the iteration procedure can be cut off (with ε) at ε^3. Then the Langevin type equation for the amplitude W will take the form

$$C_0\partial_\tau W = [C_1(R - R_c) - C_2|W|^2]W + C_3\partial_\xi^2 W - iC_4\partial_\xi^1\partial_\eta^2 W - C_5\partial_\eta^4 W + 2\gamma(\xi, \eta, \tau). \tag{12}$$

Here, the dimensionless coefficients C_i are expressed through the nematic parameters and have a cumbersome form (see [3]). The term γ in equation (12) describes random forces whose correlator can be found in a standard way.

From equation (12) it is possible, in the known manner (see [2]), to change over to the Fokker–Planck equation for the probability density of distribution of amplitudes W, W^*, and from the latter, by following [2], it is possible to reconstruct the appearance of the functional which we are interested in; this describes in the predetermined approximation the nematic liquid crystal properties near (but above) the threshold of the electrohydrodynamic instability. The functional, written in the initial

variables x, y, z, t, has the form

$$\mathscr{F} = \iint dx\mathrm{d}y[-(R - R_c)C_1|W|^2 + C_2|W|^4 + C_3|\partial_x W|^2 + iC_4(\partial_x W\partial_y^2 W^* - \partial_x W^*\partial_y^2 W) + C_5|\partial_y^2 W|^2], \tag{13}$$

where we take into account the fact that, according to [3], C_4 is a ($C_4 \sim k_0$) pseudoscalar quantity. Consequently, the functional is invariant with respect to the substitution of X for $-X$ and k_0 for $-k_0$.

Calculations performed in [3] for the case of 4-methoxybenzylidene-4′-butylaniline (MBBA) demonstrate that numerical values of the coefficients for the two first terms in equation (13) are much greater than those of the remaining terms (containing derivatives). This observation allows to consider in the long wavelength approximation the spatial change occurs without a change of the module (see e.g. the principle of module conservation in [4]). In view of this, the following substitution is expedient:

$$\left.\begin{aligned} W &= |W_0| \exp(i\varphi), \\ W^* &= |W_0 \exp(-i\varphi), \end{aligned}\right\} \qquad |W_0| \approx \text{const.}, \tag{14}$$

assuming

$$\varphi \equiv k_0 X(x, y), \tag{15}$$

where k_0 is the wavevector of the initial system of rolls which is modulated along the OX axis. This substitution leads the functional (13) to assume the form:

$$\Delta\mathscr{F} = k_0^2|W_0|^2 \iint dxdy[C_3(\partial_x X)^2 + 2k_0 C_4(\partial_x X)(\partial_y X)^2 + C_5(k_0^2(\partial_y X)^4 + (\partial_y^2 X)^2)]. \tag{16}$$

In such a form, the functional (16) coincides with respect of its form with the contribution of elastic deformations to the free energy of a smectic A liquid crystal in the case where the dependence of the elastic deformations of the Z variable is not taken into account.

In the general case, in the functional under consideration, a term of the form $C_6(\partial_y X)^2$ should be present, which describes the anisotropy of the energy of rolls. A corresponding addition to the value of free energy will appear in the case of a transition from the three-dimensional functional to that for the quasi-two-dimensonal by averaging the functional over the plate thickness, taking into account the hard boundary conditions for the director. Given this the generalised thermodynamic potential of the system can be finally written as

$$\Delta\mathscr{F} = k_0^2|W_0|^2 \iint dxdy[C_3(\partial_x X)^2 + 2k_0 C_4(\partial_x X)(\partial_y X)^2 + C_5(k_0^2(\partial_y X)^4 + (\partial_y^2 X)^2) + C_6(\partial_y X)^2]. \tag{17}$$

3. Incommensurate structures

The presence of the generalized thermodynamic potential term in the C_4 coefficient may result in additional features in the behaviour of the roll system. In particular, there is a significant possibility of a helical torsion of the rolls. There then arises the

question of the possibility of observing this phenomenon. In the recently published paper [1], this effect was experimentally observed.

In the present study, a standard scheme has been utilized for obtaining the Williams domains, this scheme has a slight modification which consists in that to one of the plates an additional voltage is applied, spatially modulated in the OX direction, which exceeds the main voltage by 0·6 V. We have observed a deformation in a system of rolls, whose interpretation we propose to carry out with the use of a functional which had been obtained for the purpose of describing the Benard effect in an isotropic liquid. At the same time, it is recognized that the question of explaining the formation of the observed structures remains unsolved. As we can see from figure 3 in [1], the deformed system has changed from one-dimension to two-dimensions, this fact cannot be explained within the framework of the proposed functional. For the purpose of an adequate description of the experimental situation with the use of the generalized thermodynamic potential obtained, one more term should be taken into account, which appears because of the application of an additional periodic voltage to the plate. The form of this term can be obtained in the following way.

Application, apart from a constant field, of a weak periodic electric field, should, generally speaking, be obtained anew. Accounting for the influence of this spatially-periodic field can however be carried out within the framework of the functional obtained. Actually, since the period of the applied field is of the same order of the period of repetition of the rolls and is much less than a typical size of an inhomogeneity, which can be described by our functional (the system of rolls in a continual approximation), only the long wavelength contribution will be substantial. Such a contribution results because of interference which is associated with the closeness, rather than the coincidence of the periods of the applied external and modulated self-coordinated internal electric fields. Such a method of accounting for the contribution of interference to the long wavelength part of the permittivity is well-known from the theory of solids (see, e.g., [5]).

The most substantial contribution of the electric field to the functional (13) is contained in the R term (since $R \sim E_z^2$, where E_z is the component of the electric field along the OZ axis). The intensity E_z can be presented as a sum of the three terms:

$$E_z = E_0 + E_{\text{in}} \cos(k_0 x + \varphi(x)) + \Delta E \cos(k_1 x), \tag{18}$$

where E_0 is the external electric field, ΔE is the spatially modulated additional field, E_{in} is the self-coordinated internal field which arises in the nematic through the redistribution of charges; moreover, k_0 is the wavevector which characterizes an initial (prior to the application of ΔE) distribution of rolls, k_1 is the wavevector of an applied modulated addition, and finally $\varphi(x)$ is a certain phase which is to be determined (this is slowly changing at a distance of the order k_0^{-1}).

In the experimental study, the external-field potential, V is 6 V, the amplitude of the modulating addition, ΔV is 0·6 V,

$$k_1 = \frac{2\pi}{200}\,\mu\text{m}, \quad \frac{2\pi}{168} \leqslant k_0 \leqslant \frac{2\pi}{232}.$$

In this range of parameters, a two-dimensional pattern of rolls has been observed. In the case where k_0 was outside this range, the one-dimensional system of rolls was realized.

As we have been noted, only long wavelength components would be legitimate to be left in the functional (13), which is achieved by averaging this functional according

to one of the $(2\pi/k_0$ or $2\pi/k_1)$ periods. It follows, however, from the form of this functional that only the terms containing E_z is averaged. The result of this averaging depends essentially on the ratio of periods, k_0/k_1. In the experiment [1], $-0{\cdot}16 \leqslant (k_0 - k_1)/(k_1) \leqslant 0{\cdot}16$.

Let us write E_z^2 in the form

$$E_z^2 = E_0^2 + E_{in}^2 \cos[(k_0 - k_1)x + \varphi(x)] + \cdots \tag{19}$$

Here, the points indicate terms which vanish on averaging. In so doing, we have made use of the identity

$$2\cos(k_0 x + \varphi(x)) \cdot \cos k_1 x = \cos[(k_0 - k_1)x + \varphi(x)] + \cos[(k_0 + k_1)x + \varphi(x)]$$

and have omitted the second term which does not permit a contribution to the long wavelength asymptotes.

Now we must only carry out practical averaging of the last term in equation (13). In this connection, we shall pay attention to the interesting peculiarities of a structure which was observed in [1]. In the experiment, the k_0 wavevector was changing (with fixed k_1) within the range

$$\frac{1}{2} < \frac{k_0}{k_1} < \frac{3}{2}.$$

These peculiarities consist of the fact that the two-dimensional structures were observed over the whole band of variation of the wavevector, k_0, except the δ-neighborhood $(\delta \approx 0{\cdot}16k_1)$ of ratios between the periods

$$\frac{k_0}{k_1} = \frac{1}{2}, \quad \frac{k_0}{k_1} = \frac{3}{2}.$$

We shall now consider in more detail the behaviour of the system near these ratios, k_0/k_1.

Let us assume $k_0 = k_1 + \Delta_0$, where Δ_0 is a small desynchronization of the phases. In this case, taking this into account with the relationship (19), our functional takes the form:

$$\begin{aligned}\mathcal{F} = \iint dxdy[&C_1|W|^2(R_c - R) - C_1|W|^3 R_c \mu \frac{\Delta E}{E_0} \\ &\times \cos(\Delta_0 x - \varphi(x)) + C_2|W|^4 + C_3|\partial_x|W|^2 + iC_4 \\ &\times (\partial_x W \partial_y^2 W^* - \partial_x W^* \partial_y^2 W) + C_5|\partial_y^2 W|^2 + C_6|\partial_y W|^2].\end{aligned} \tag{20}$$

In this expression we have expressed the internal field, E_{in}, through the modulus of the order parameter, $|W_0|$, in accord with the relationship (see [3]):

$$E_{in} = |W_0|\mu E_0 \tag{21}$$

where μ is determined by the nematic parameters.

Estimates carried out in [3] for the MBBA case have demonstrated that numerical values of coefficients C_1 and C_2 are much greater than the remaining coefficients of the gradient terms. This circumstance allows to consider that in the long wavelength approximation, the spatial change of W takes place without module variation (see, e.g., the principle of module conservation [4]), and to look for a solution which

realizes the extremum of the functional in equation (20) of the form

$$\left.\begin{aligned} W &= |W_0| \exp(i\varphi(x) + ik_y y), \\ W^* &= |W_0| \exp(-i\varphi(x) - ik_y y), \end{aligned}\right\} W_0 = \text{const.} \tag{22}$$

As a result of the Euler equation for the functional (20), we can write:

$$B\partial_x \Psi + 2\alpha \sin\Psi \cos\Psi = 0. \tag{23}$$

Here, for the purpose of convenience for further comparisons with the study [6], the following notations are introduced

$$\Psi = \varphi(x)/2, \quad B = 8C_3 W_0^2, \quad \alpha = C_1 r_c \mu W_0^2 \Delta E/E_0 \tag{24}$$

The first integral of equation (23) is easily found:

$$B(\partial_x \Psi)^2/2 + \alpha \sin^2\Psi = \text{const.} \equiv \frac{\alpha}{\kappa^2} \tag{25}$$

In contrast to [6], the parameter κ introduced contains the parameter k_y^2 which has not yet been determined. The value of Ψ and consequently the period of a structure along the OX axis can easily be found from equation (25).

$$l_x = 4\kappa K(\kappa)(B/2\alpha)^{1/2}, \tag{26}$$

where $K(\kappa)$ is the complete elliptic integral of the first kind.
The functional (20), referred to unit area in the notation of equation (24), to an approximation to constant values containing no variables, will take the form

$$\bar{\mathcal{F}} = 1/l_x \int_0^{l_x} dx\, [B(\partial_x \Psi)^2/2 - b\partial_x \Psi - \alpha \sin^2\Psi + C_6 W_0^2 k_y^2 + C_5 W_0^2 k_y^4], \tag{27}$$

where

$$b = 4C_4 W_0^2 k_y^2 + 8\Delta_0 C_3 W_0^2,$$

denoting $b_0 = 8\Delta_0 C_3 W_0^2$, we shall write $b = b_0 + 4C_4 W_0^2 k_y^2$.

The value of the functional (27) on the extreme of equation (23) is

$$\bar{\mathcal{F}} = \frac{\pi b\sqrt{(2\alpha/B)}}{2\kappa K(\kappa)} - \alpha/\kappa^2 + 2\alpha E(\kappa)/\kappa^2 \cdot K(\kappa) + C_5 W_0^2 k_y^4 + C_6 W_0 k_y^2 + \alpha/2. \tag{29}$$

Here, $E(\kappa)$ is the complete elliptic integral of the second kind.

Parameters κ and k_y^2, which are still unknown, should be determined from the condition of the minimum of equation (29)

$$\bar{\mathcal{F}}'_{k_y^2} = -(2\pi C_4 W_0^2 (2\alpha/B)^{1/2})/\kappa K(\kappa) + 2C_5 W_0^2 k_y^2 + C_6 W_0^2 = 0, \tag{30}$$

$$\begin{aligned} \bar{\mathcal{F}}'_{\kappa} = {} & -2\alpha E^2(\kappa)/\kappa^3(1-\kappa^2)K^2(\kappa) + (4\pi C_4 W_0^2 k_y^2 + b_0) \\ & \times ((2\alpha/B)^{1/2} \cdot E(\kappa))/2\kappa^2(1-\kappa^2)K^2(\kappa) = 0. \end{aligned} \tag{31}$$

By substituting values of k_y^2 from equations (30) to (31), we obtain an equation which determines the κ parameter

$$\frac{E(\kappa)}{\kappa} - \frac{b_0}{2\sqrt{(2\alpha B)}} = \frac{4\pi C_4 W_0^2}{BC_5} \cdot \left[\frac{\pi}{2\kappa K(\kappa)} - \frac{C_6}{4C_4}\sqrt{\left(\frac{B}{2\alpha}\right)}\right] \tag{32}$$

The conditions for the fact that the functional (29) at the extreme point takes the minimum value

$$\mathcal{F}_{\kappa}'' \geqslant 0, \quad \mathcal{F}_{k_y^2}'' \geqslant 0, \quad \mathcal{F}_{\kappa}'' \cdot \mathcal{F}_{k_y^2}'' - \mathcal{F}_{\kappa k_y^2}''^2 \geqslant 0$$

are reduced in our case to the following relationship:

$$K^3(\kappa)(1 - \kappa^2)/E(\kappa) \geqslant 2\pi^2 C_4^2 W_0^2/BC_5. \tag{33}$$

Here, κ is the radical of equation (32). If the combination of parameters, included into the right hand side of the inequality (33), is determined from equation (32), then instead of equation (33) we can write

$$\frac{K^2(\kappa)(1 - \kappa^2) - E^2(\kappa)}{\kappa E(\kappa)} \geqslant \frac{1}{2\sqrt{(2\alpha B)}} [2\pi W_0^2 C_4 C_6/C_5 - b_0]. \tag{34}$$

Since the left hand side is smaller or equal to zero, the condition of existence of the minimum of the functional (29) takes the form:

$$b_0 \geqslant 2\pi W_0^2 C_4 C_6/C_5. \tag{35}$$

Hence it can be seen that with other than zero value of the C_6 coefficient which characterizes the energy of roll anisotropy, a certain mismatching of phases is required for the appearance of two-dimensional structures

$$\Delta_0 \geqslant \pi C_4 C_6/4C_3 C_5. \tag{36}$$

Since, however, we know nothing about the possible value of the coefficient C_6, it is difficult to say whether the observed peculiarity would be noted experimentally.

We now demonstrate that the process of forming the two-dimensional structures has a threshold character by α, i.e., by an applied modulated electric field. Really, the prerequisite for the formation of two-dimensional structures consists in the presence of such solutions to equation (32), with which

$$k_y^2 \geqslant 0.$$

Therefore, let us first determine the value $\alpha = \alpha_0$, with which the condition $k_y^2 = 0$ is compatible with equation (32). Then it follows from equation (32)

$$\frac{E(\kappa_0)}{\kappa_0} = \frac{b_0}{2\sqrt{(2\alpha_0 B)}}, \quad \frac{\pi}{2\kappa_0 K(\kappa_0)} = \frac{C_6}{4C_4}\sqrt{\left(\frac{B}{2\alpha_0}\right)}. \tag{37}$$

Hence, excluding α_0, we obtain the equation which determines κ_0

$$K(\kappa_0)E(\kappa_0) = \frac{\pi\Delta_0 C_4}{C_6}. \tag{38}$$

This equation always has a solution with

$$\frac{\Delta_0 C_4}{C_6} \geqslant \frac{\pi}{4}. \tag{39}$$

It can easily be seen that the criterion (39) is compatible with (36) if $C_4^2 \geqslant C_3 C_5$. Estimates for MBBA demonstrate [3] that the latter inequality is satisfied with a great margin. Substitution of the value of κ_0, found from the solution (38), into one of equations (37), will determine the critical value of α_0.

Now let us find the form of k_y^2 near the threshold (α_0, κ_0)

$$k_y^2(\alpha_0 + d\alpha, \kappa_0 + d\kappa) \approx \left(\frac{\partial k_y^2}{\partial \kappa}\right)_{\kappa=\kappa_0} \cdot d\kappa + \left(\frac{\partial k_y^2}{\partial \alpha}\right)_{\alpha=\alpha_0} \cdot d\alpha$$

$$= \left(\frac{\partial k_y^2 d\kappa}{\partial \kappa d\alpha} + \frac{\partial k_y^2}{\partial \alpha}\right)_{\kappa=\kappa_0, \alpha=\alpha_0} \cdot d\alpha. \tag{40}$$

The dependence of κ on α is determined by equation (32). Using this equation, we obtain after some transformations:

$$\frac{1}{2\alpha}\frac{d\alpha}{d\kappa} = \frac{1}{\kappa(1-\kappa^2)}\left[\frac{E(\kappa)}{K(\kappa)} - \frac{E^2(\kappa) - (1-\kappa^2)K(\kappa)}{E(\kappa)K(\kappa) - 2\pi^2 C_4 W_0^2/BC_5}\right].$$

Substitution of this relationship into equation (40), accounting for equations (37) and (32), permits the representation of k_y^2 as

$$k_y^2 = \frac{C_4}{C_5}\sqrt{\left(\frac{2}{\alpha_0 B}\right)}\frac{\pi\Lambda d\alpha}{2\kappa_0 K(\kappa_0)} = \Lambda\frac{C_6 d\alpha}{4C_5\alpha_0}, \tag{41}$$

where

$$\Lambda \equiv \frac{E^2(\kappa_0) - (1-\kappa_0^2)K^2(\kappa_0)}{E^2(\kappa_0) - (1-\kappa_0^2)K^2(\kappa_0) - \dfrac{\kappa_0 E(\kappa_0)}{2\sqrt{(2\alpha_0 B)}}[b_0 - 2\pi C_4 C_6 W_0^2/C_5]}. \tag{42}$$

Due to the condition (34), the parameter $\Lambda > 0$.

In [5] a situation is considered where the coefficient C_6 is equal to zero; for this case, no threshold in α appeared. At the same time, in our case, as we can see from equation (41), the two-dimensional structure occurs only with $\alpha > \alpha_0$.

For the convenience of comparison with the study in [7], we represent k_y^2 in the following form, using the first equation (30):

$$k_y^2 = \frac{C_4}{2C_5}\left[\frac{2\pi}{\kappa K(\kappa)}\sqrt{\left(\frac{2\alpha}{B}\right)} - \frac{C_6}{C_4}\right]. \tag{43}$$

Hence, with $C_6 \to 0$ and $\kappa \to 1$, there follows the result of the study [7]. The dependence between the observed magnitudes is of interest to experimentalists. Such a dependence follows from equation (43), taking into account equations (36) and (37),

$$k_y^2 = \frac{4\pi C_4}{C_5}\left[\frac{1}{l_x(\alpha)} - \frac{1}{l_x(\alpha_0)}\right]. \tag{44}$$

Let us consider two-dimensional structures with $k_0 = 3k_{1/2} + \Delta_0$.

As we have seen, two-dimensional structures had not been observed with k_0 close to $k_1/2$ and $3k_1/2$. Now let us consider in more detail the situation for k_0 close to $3k_1/2$. We assume

$$k_0 = 3k_1/2 + \Delta_0, \quad \Delta_0 \ll k_0 \tag{45}$$

In contrast to the case where $k_0 = k_1 + \Delta_0$, now the second term in equation (18), equal to

$$E_{in}\Delta E \cos[k_1 x/2 + \Delta_0 x + \varphi(x)],$$

will be rapidly oscillating function because of the presence of the $k_1 x/2$ term in the cos argument. This term should be averaged over the $I_1 = 4\pi/k_1$ period. Naturally,

it is to be taken into consideration that although the phase $\varphi(x)$ changes slightly during one period within the limits of an infinite system, such a change of the phase will be substantial. By averaging, e.g. over the nth period, we have (with $\Delta_0 l_1 \ll 1$)

$$\frac{1}{l_1}\int_{nl_1}^{(n+1)l_1} \cos((2\pi/l_1 + \Delta_0)x + \varphi(x))dx \approx (\Delta_0 l_1 + \varphi'(x_n)/2\pi \times \cos(\Delta_0 x_n + \varphi(x_n)), \quad x_n \equiv nl_1 \tag{46}$$

(In the continual approximation, $x_n \to x$.) While we can neglect $\varphi'(x_n)$ in equation (46), in the given case with an obvious change of notation $\alpha \to (2\Delta_0/k_1) \cdot \alpha$, we come again to the functional (27). Since $2\Delta_0/k_1 \ll 1$, a critical field in which two-dimensional structures are observed, in the case when $k_0 \approx \frac{3}{2}k_1$ will be the same number of times greater than the critical field, with $k_0 \approx k_1$. As can be easily understood, a similar situation will also take place with $k_0 \approx k_1/2$. Consequently, near the rational relationships

$$\frac{k_0}{k_1} = \frac{1}{2}, \frac{3}{2}$$

the critical field will be inversely proportional to Λ_0, while when $k_0 \approx k_1$ this field depended only on the system parameters. This constitutes a principal difference of system behaviour near $k_0 \approx k_1$ and

$$k_0 \approx \tfrac{1}{2}k_1, \tfrac{3}{2}k_1$$

Experimentally, this peculiarity should be revealed in the following way. With such a preset voltage of a simulated field ΔE that two-dimensional structures occur with $k_0 \approx k_1$, it is always possible to choose a sufficiently small region of the wavevector values near $k_1/2$ and $3k_1/2$, inside which an applied field will be less than critical. Consequently, in these small regions the two-dimensional structures will not occur. A similar picture has been noted in [1].

Now let us briefly discuss the possibility of neglecting the $\varphi'(x)$ derivative in equation (46). The first condition of such a neglect will evidently be

$$\Delta_0 l_1 \geqslant \varphi'(x), \tag{47}$$

which is reduced to

$$\Delta_0 l_1 \gg (2\alpha/B\kappa^2)^{1/2} \cdot (1 - \kappa^2)^{1/2}. \tag{48}$$

The condition in equation (48) results from the demand of the smallness of $\varphi' \cos(\Delta_0 x + \varphi(x))$ as compared to the previous terms:

$$\Delta_0 > \frac{\alpha}{B}. \tag{49}$$

All of this relates to the case when W_0 is constant. It is possible to demonstrate that within the framework of the set problem, breakdowns in the role system may also take place. Actually, despite the possibility of the existence of various physical mechanisms leading to breakdown in the roll system, which have been observed experimentally even within the framework of our functional (and with the assumption that we leave only phase derivatives), we can demonstrate that the breakdowns inevitably occur under certain relationships of parameters.

Accounting for spatial derivatives leads to re-normalization of R_c, and with sufficiently high values of derivatives in some points of the domains, the difference $R_c - R$ may tend to zero, which fact indicates a breakdown in these points of the domains. Although the derivatives by W_0 should be successively taken into account as well, we believe our evaluations to be correct at last to an order of magnitude. To demonstrate this, we return to the functional (20), having substituted the solutions in equation (22) with $W_0 \neq \text{const}$, and having retained only phase derivatives we find

$$\mathcal{F} = \iint dxdy \left\{ [C_1(R_c - R) + C_3(\partial_x \varphi)^2 + 2C_4 k_y^2 \partial_x \varphi + k_y^2 C_6 + k_y^4 C_5] W_0^2 - C_1 \mu \frac{\Delta E}{E_0} \cdot \cos(\Delta_0 x + (\varphi(x)) W_0^3 + C_2 W_0^4 \right\}. \tag{50}$$

In a long wavelength approximation with $\kappa \to 1$, the solution to $\varphi(X)$ is known

$$\varphi(x) = \pi/2 + 2\,\text{arctg}\left(\exp\left[2\sqrt{\left(\frac{\alpha}{2B}\right)}(x - l_x/4)\right]\right). \tag{51}$$

By substituting this asymptotic value into the Euler equation for the functional (50) (the unknown amplitude, W_0, is varied), we find that a breakdown occurs when the system parameters satisfy

$$2W_0 \cdot \left[C_1(R_c - R) + \frac{4\pi C_4}{C_5} \cdot \left(\frac{1}{l_x} - \frac{1}{l_0}\right)\left(4\pi C_4\left(\frac{1}{l_x} - \frac{1}{l_0}\right) + C_6\right)\right] + 3W_0^2\left[4\beta^2 C_3 - C_1 R_c \mu \frac{\Delta E}{E_0}\cos(\Delta_0 x + \varphi(x))\right] + W_0^{3/2} \cdot 40\pi \frac{C_4^2}{C_5}\beta \cdot \left(\frac{1}{l_x} - \frac{1}{l_0}\right) + 4W_0^3 C_2 = 0, \tag{52}$$

where

$$l_x \equiv l_x(\alpha), \quad l_0 \equiv l_x(\alpha_0).$$

Changing to the variables $w = W_0^{1/2}$, we write this equation in the form

$$w^2(w^4 + pw^2 + qw + r) = 0, \tag{53}$$

where

$$\begin{aligned} p &= \frac{3}{4C_2}\left(4\beta^2 C_3 - C_1 R_c \mu \frac{\Delta E}{E_0}\cos(\Delta_0 x + \varphi(x))\right), \\ q &= 10\frac{\pi C_4^2}{C_2 C_5} \cdot \beta \cdot \left(\frac{1}{l_x} - \frac{1}{l_0}\right), \\ r &= \frac{1}{4C_2}\left(C_1(R_c - R) + \frac{4\pi C_4}{C_5}\left(\frac{1}{l_x} - \frac{1}{l_0}\right)\left(C_6 + 4\pi C_4\left(\frac{1}{l_x} - \frac{1}{l_0}\right)\right)\right). \end{aligned} \tag{53'}$$

In the point of roll breakdown, $w = 0$. We consider the solution to equation (53) near $r \approx 0$ and, consequently, near the solution $w \approx 0$. In this case, we can neglect the term w^4 in the equation (53), and the resulting equation

$$pw^2 + qw + r = 0 \tag{54}$$

will have the solutions

$$w_1 = -\frac{q}{p} \quad \text{and} \quad w_2 = \frac{r}{q}. \tag{55}$$

Only the second solution however realizes the minimum of the functional (50). It means physically that as the value of l_x decreases, a moment will come when in some points, according to the expression of r from equation (53), $r = 0$ and, consequently, either w or W_0 also tend to zero, which fact indicates breakdown in a roll system.

Thus, within the framework of our approach we can explain not only the formation of one- and two-dimensional structures but also the presence of periodical breakdown in a roll system, which have been observed experimentally [1].

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